

Entanglement Entropy of the Low-Lying Excited States and Critical Properties of an Exactly Solvable Two-Leg Spin Ladder with Three-Spin Interactions

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In this work, we investigate an exactly solvable two-leg spin ladder with three-spin interactions. We obtain analytically the finite-size corrections of the low-lying energies and determine the central charge as well as the scaling dimensions. The model considered in this work is in the same universality class of critical behavior of the XX chain with central charge $c = 1$. By using the correlation matrix method, we also study the finite-size corrections of the Rényi entropy of the ground state and of the excited states. Our results are in agreement with the predictions of the conformal field theory.

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I. INTRODUCTION

Perhaps one of the most popular and precise ways to determine the critical behavior of one-dimensional quantum systems is through the finite-size corrections of the energy. The machinery of conformal field theory (CFT) establishes that the ground state energy of a one-dimensional critical system of size L , under periodic boundary condition (PBC), behaves asymptotically as¹⁻³

$$\frac{E_0}{L} = e_\infty - \frac{v_s \pi c}{6L^2} + o(L^{-2}), \quad (1)$$

where v_s is the sound velocity, e_∞ is the bulk ground state energy per site, and c is the central charge.

The mass gap amplitudes of the finite-size corrections of the higher energy states, are related with the scaling dimensions d^β . There are, for each primary operator O_β ($\beta = 1, 2, \dots$) in the CFT, a tower of states $E_{j,j'}^\beta(L)$ in the spectrum of the Hamiltonian with asymptotic behavior^{3,4}

$$E_{j,j'}^\beta(L) - E_0(L) = \frac{2\pi v_s}{L}(d^\beta + j + j') + o(L^{-1}), \quad (2)$$

where $j, j' = 0, 1, 2, \dots$. The above relations were used systematically to determine the universality class of critical behavior of several models with great success. The success of these relations resides in the fact that it is possible to obtain the critical exponents, which are associated with physical quantities in the thermodynamic limit, from *finite* systems.

Recently, a great deal of excitement has been generated due to the fact that the universality classes of critical behavior of one-dimensional systems can also be inferred from the finite-size corrections of the entanglement entropy of the ground state as well as of the excited states.⁵⁻⁹ Below, we briefly report some important results about these corrections that we will explore in this work.

Consider a quantum chain with L sites, described by a pure state whose density operator is ρ . Let us consider

that the system is composed by the subsystems \mathcal{A} with ℓ sites ($\ell = 1, \dots, L-1$) and \mathcal{B} with $L-\ell$ sites. The Rényi entropy is defined as

$$S_\alpha(L, \ell) = \frac{1}{1-\alpha} \ln \text{Tr}(\rho_{\mathcal{A}}^\alpha), \quad (3)$$

where $\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}} \rho$ is the reduced density matrix of the subsystem \mathcal{A} . The von Neumann entropy, also known as entanglement entropy, is the particular case $\alpha = 1$. In the scaling regime $1 \ll \ell \ll L$, it is expected that for the critical one-dimensional systems with PBC, the Rényi entropy of the ground state behaves as

$$S_\alpha(L, \ell) = S_\alpha^{CFT}(L, \ell) + S_\alpha^{osc}(L, \ell). \quad (4)$$

The first term in this equation, is the CFT prediction and is given by^{8,10-12}

$$S_\alpha^{CFT} = \frac{c}{6} \left(1 + \frac{1}{\alpha} \right) \ln \left[\frac{L}{\pi} \sin \left(\frac{\pi \ell}{L} \right) \right] + c_\alpha, \quad (5)$$

where c_α is a non-universal constant. The second term is given by¹³⁻¹⁵

$$S_\alpha^{osc} = \frac{[a_1 \delta_{1,\alpha} + g_\alpha (1 - \delta_{1,\alpha}) \cos(\kappa \ell + \phi)]}{L^{p_\alpha}} \left| \sin \left(\pi \frac{\ell}{L} \right) \right|^{-p_\alpha}, \quad (6)$$

where p_α is a new critical exponent, and a_1 and g_α are non-universal constants. The wave vector κ and the phase ϕ depend on the model. For instance, for the Ising model $\kappa = 0 = \phi$, and for the spin- s XXZ chains at zero magnetic field $\kappa = \pi$ and $\phi = 0$. For the systems with PBC, it is not expected oscillations in the von Neumann entropy S_1 .^{13,16} The origin of the exponent p_α in the above equation are the conical space-time singularities produced in the conformal mapping used to describe the reduced density matrix $\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}} \rho$ in the CFT.¹⁷ This exponent is related to the scaling dimension x^{con} of

an operator (unknown) of the underlying CFT as follows $p_\alpha = 2x^{con}/\alpha$.¹⁷ There are evidences that the exponent $x^{con} = x_\epsilon$ for $\alpha > 1$, where x_ϵ is the scaling dimension of the energy operator, and that $p_1 = \nu = 2$ for all models.¹⁶ On the other hand, the origin of the oscillating factor $[\cos(\kappa\ell + \phi)]$ is not yet completely understood, however it has been observed in systems with anti-ferromagnetic tendencies.

The CFT prediction for the finite-size corrections of the Rényi entropy above were verified by several authors. Both the central charge^{13–15,18–21} and the exponent p_α ^{13,15,16,22,23,26} were obtained with amazing precision.

The Rényi entropies of the excited states also present very interesting universal behavior.^{9,24,25} The α -Rényi entropy S_α^Υ , associated with an excitation of a primary operator Υ with conformal weight h , is related with the 2α -point correlator of the operator Υ and its conjugate by^{9,24}

$$\ln F_\Upsilon^{(\alpha)} = (1 - \alpha)[S_\alpha^\Upsilon - S_\alpha^{gs}],$$

where S_α^{gs} is the α -Rényi entropy of the ground state and

$$F_\Upsilon^{(\alpha)} = \alpha^{-2\alpha(h+\bar{h})} \frac{\langle \prod_{j=0}^{\alpha-1} \Upsilon[2\pi j/\alpha] \Upsilon^\dagger[2\pi(j + \ell/L)/\alpha] \rangle}{\langle \Upsilon[0] \Upsilon^\dagger[2\pi\ell/L] \rangle^\alpha}.$$

In particular, $F_\Upsilon^{(\alpha)} = 1$ for the operators associated with compact excitations,^{9,24} and the Rényi entropy of these excitations are the same of the ground state, i. e., $S_\alpha^\Upsilon = S_\alpha^{gs}$. On the other hand, for non-compact excitations the universal function $F_\Upsilon^{(\alpha)}$ depends on the particular primary operator Υ . For instance, for the primary operator $\Upsilon = i\partial\phi$ of a free boson theory the universal function $F_{i\partial\phi}^{(2)}$ was calculated analytically and is given by^{9,24}

$$F_{i\partial\phi}^{(2)} = 1 - 2s^2 + 3s^4 - 2s^6 + 8s^8 \quad (7)$$

where $s = \sin(\pi\ell/2L)$.

In this work, we intend to explore the above relations in order to investigate a two-leg spin ladder model, which is exactly soluble. The paper is organized as follows. In Sec. II, we present the model and its ground state phase diagram. In Sec. III, we determine the Rényi entropy of the model by using the correlation matrix method. Finally, in Sec. IV we summarize our results.

II. PHASE DIAGRAM AND CRITICAL PROPERTIES OF A TWO-LEG SPIN MODEL

We consider the following Hamiltonian defined on a two-leg geometry

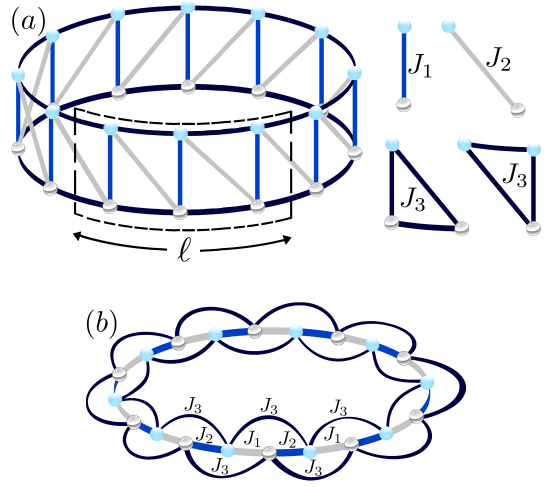


Figure 1: (Color online). (a) Schematic representation of the two-leg spin ladder. J_1 , J_2 and J_3 are the couplings between the spins along the rung, the diagonal and the plaquettes, respectively. We considered a bipartite system as reported in the figure. The system of L rungs is divided into two subsystems of sizes ℓ and $(L - \ell)$. (b) The spinless Hamiltonian (9). The lines represent the hopping terms.

$$H = 2 \sum_{\beta=x,y} \sum_{\lambda=1}^2 \sum_{j=1}^L (J_\lambda s_{\lambda,j}^\beta s_{\lambda+1,j+\lambda-1}^\beta +$$

$$J_3 s_{\lambda,j}^\beta s_{\lambda+1,j+\lambda-1}^z s_{\lambda,j+1}^\beta) - h \sum_{\lambda=1}^2 \sum_{j=1}^L s_{\lambda,j}^z, \quad (8)$$

where J_j ($j = 1, 2, 3$) are coupling constants [see Fig. 1(a)], $s_{\lambda,j}^\beta$ ($\beta = x, y, z$) are the spin- $\frac{1}{2}$ operators at leg $\lambda = 1, 2$ and rung j , h is the magnetic field, and L is considered even. We investigate the above model with PBC, i. e., $s_{\lambda,j+L}^\beta = s_{\lambda,j}^\beta$ and $s_{\lambda+2,j}^\beta = s_{\lambda,j}^\beta$. Note that this model has three-spin interactions. Models with multi-spin interactions like $s_j^\beta \prod_{l=j+1}^m s_l^{\beta'}$ (with $\beta, \beta' = x, y$) can be mapped into a fermionic quadratic form, as first noted by Suzuki.²⁷ In the particular case of three-spin interactions, several variants of the Hamiltonian (8) were considered in the literature^{28–34} (see also Ref. 35). It is quite interesting to mention that spin- $\frac{1}{2}$ Hamiltonians with three-spin interactions can be generated using optical lattices.³⁶

The authors of Ref. 32 also investigated the same model considered in this work. However, these authors focused in the effect of the lattice distortion in the infinity system. Here, our emphasis is in the *finite-size corrections* (i) of the entanglement entropy and (ii) of the low-lying energy states in the critical regions. As we already mentioned, these corrections are related with the central charge and the scaling dimensions.

For the sake of clarity, we briefly describe the procedure to diagonalize the Hamiltonian (8) exactly.

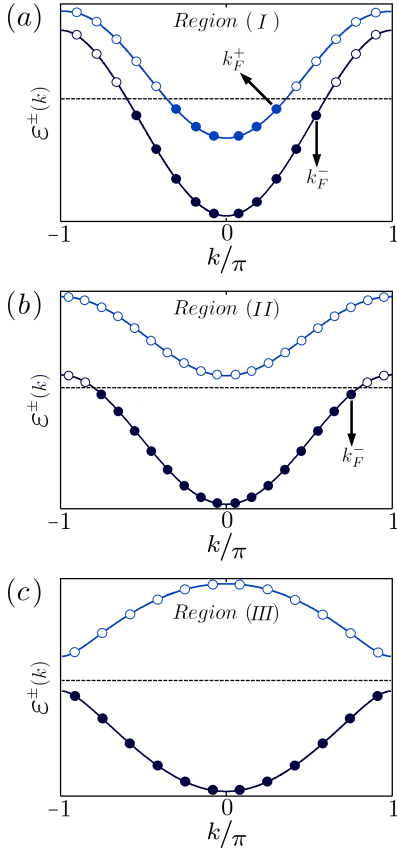


Figure 2: (Color online). Figures (a), (b) and (c) display the three typical profiles of the band dispersion, for $h = 0$, that appear in the regions I, II and III, respectively [see Fig. (3)]. k_F^\pm are the Fermi momenta in the branches $\sigma = \pm$ (see text).

First, we use a Jordan-Wigner transformation [$c_m = (s_m^x - i s_m^y) \prod_{j < m} (-2s_j^z)$] in order to map the Hamiltonian (8) in a spinless fermion chain with nearest-neighbor and next-neighbor hoppings [see Fig. 1(b)]. By using this transformation we obtain

$$H = \sum_{i=1}^{L-1} \left(J_1 c_{2i-1}^\dagger c_{2i} + J_2 c_{2i}^\dagger c_{2i+1} \right) - \sum_{j=1}^{2L-2} \frac{J_3}{2} c_i^\dagger c_{i+2} + J_1 c_{2L-1}^\dagger c_{2L} - e^{i\pi \hat{N}_F} \left[J_2 c_{2L}^\dagger c_1 - \frac{J_3}{2} (c_{2L-1}^\dagger c_1 + c_{2L}^\dagger c_2) \right] + h.c. - h \hat{N}_F + hL, \quad (9)$$

where $\hat{N}_F = \sum_{i=1}^{2L} c_i^\dagger c_i$ is the particle number operator. Note that the original two-leg ladder system of size L is mapped into a chain with $2L$ sites.

Finally, by using the following Fourier transformations

$$c_{2j-1} = \sum_k \frac{e^{ikj}}{\sqrt{L}} a_k \text{ and } c_{2j} = \sum_k \frac{e^{ikj}}{\sqrt{L}} b_k, \quad (10)$$

we can express the Hamiltonian (9) in the following diagonal form

$$H = \sum_{\sigma=\pm} \sum_{\{k\}} \varepsilon^\sigma(k) d_k^{\sigma\dagger} d_k^\sigma + hL, \quad (11)$$

where $d_k^\pm = \frac{1}{\sqrt{2}} \left(a_k \pm \frac{J_1 + J_2 \exp(-ik)}{\sqrt{J_1^2 + J_2^2 + 2J_1 J_2 \cos k}} b_k \right)$, $k = k(j) = \frac{2j\pi + \phi}{L}$ with $\phi = 0$ (π) if N_F is odd (even) and $j = 0, \pm 1, \dots, \pm(L-2)/2, -L/2$. The band dispersion is separated in two branches ($\sigma = \pm$) given by

$$\varepsilon^\sigma(k) = -h - J_3 \cos k + \sigma \sqrt{J_1^2 + J_2^2 + 2J_1 J_2 \cos k}. \quad (12)$$

Let us focus in the case of a zero magnetic field. After an exhaustive investigation of the band dispersion above with $h = 0$, we note that depending on the coupling constants J_1 , J_2 , and J_3 , the profiles of the band dispersion present three distinct behaviors. The main characteristics of these profiles are depicted in Figs. 2(a)-(c). We label the coupling regions by regions I, II and III according to the Fermi momenta in the thermodynamic limit. In the region I (II) we have four (two) Fermi momenta, while in the region III the Fermi momenta always appear at $k_F^\pm = \pm(\pi - \pi/L)$. We can observe from Fig. 2 that the regions I and II are gapless, whereas the region III is gapped. In the latter case, the lowest excitation energy appears in the sector $s_{total}^z = -1$. The spin gap in this region is given by $\Delta = |J_1 - J_2| - J_3$.³⁷ Fig. 3 displays these regions in parameter space J_2/J_3 vs J_1/J_2 .

It is interesting also to note that depending on the coupling constants the magnetization of the ground state may not be zero in regions I and II, differently from the region III where the magnetization is always zero if $h = 0$. The magnetization per site is given by $\frac{m}{2L} = \frac{1}{2L} \sum_j <gs| s_j^z |gs> = \frac{k_F^+ + k_F^-}{2\pi} - 1/2$, where the Fermi momenta k_F^\pm are determined solving the equations $\varepsilon^\pm(k_F^\pm) = 0$, in the thermodynamic limit, and are given by

$$k_F^\pm = \arccos \left(\frac{J_1 J_2 - h J_3}{J_3^2} \pm \frac{\sqrt{J_1^2 J_2^2 + J_1^2 J_3^2 + J_2^2 J_3^2 - 2J_1 J_2 J_3 h}}{J_3^2} \right). \quad (13)$$

From the above equations we can see that for $h = 0$ the magnetization is not zero, in general. In Fig. 3, we also report the values of the intensity of magnetization m in the parameter space.

Let us now investigate the critical regions I and II in more detail. The ground state energy of the Hamiltonian (11) is obtained adding particles in the energy levels below zero, i. e.,

$$E_0 - hL = \sum_{|k| \leq k_F^-} \varepsilon^-(k) + \sum_{|k| \leq k_F^+} \varepsilon^+(k), \quad (14)$$

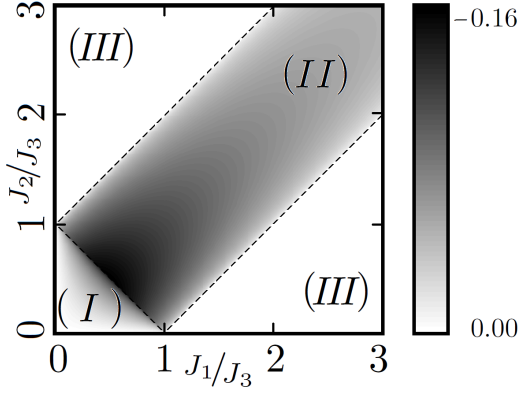


Figure 3: (Color online). The ground state phase diagram for $h = 0$. The regions I, II are gapless, whereas the region III is gapped. The right panel indicates the intensity of the magnetization per site m .

where the last sum in this equation contributes only for the region I. We consider $\mathcal{N}_F = \mathcal{N}_F^- + \mathcal{N}_F^+$ even and $k_F^\pm = (\mathcal{N}_F^\pm - 1)\pi/L$ are the Fermi momenta, where \mathcal{N}_F^σ is the particle number in the branch σ .

In order to determine the central charge c , we need to evaluate the finite-size corrections of the ground state energy [Eq. (14)]. For a fixed value of the density $\rho = \frac{\mathcal{N}_F}{2L}$ the leading finite-size correction can be obtained by using the Euler-Maclaurin formula,³⁸ which gives

$$\frac{E_0}{L} = e_\infty - \frac{\pi(v_F^+ + v_F^-)}{6L^2} + o(L^{-2}), \quad (15)$$

where $v_F^\pm = \frac{d\varepsilon^\pm}{dk}|_{k_F^\pm} = (J_3 \mp \frac{J_1 J_2}{\sqrt{J_1^2 + J_2^2 + 2J_1 J_2 \cos(2\pi\rho^\pm)}}) \sin(2\pi\rho^\pm)$ are the sound velocities associated with each branch, $\rho^\pm = \frac{\mathcal{N}_F^\pm}{2L}$, and $e_\infty = e_\infty^+ + e_\infty^-$ is the bulk energy with e_∞^\pm given by

$$e_\infty^\pm = h \left(\frac{1}{2} - 2\rho^\pm \right) - \frac{J_3}{\pi} \sin(2\pi\rho^\pm) \pm \int_0^{2\pi\rho^\pm} \frac{dk}{\pi} \sqrt{J_1^2 + J_2^2 + 2J_1 J_2 \cos k}. \quad (16)$$

If we compare Eqs. (1) and (15), we identify the central charge as $c = 1$ for each gapless mode. This value of c implies that the universality class of critical behavior of the Hamiltonian (8) is the same of the well known XX chain. Note that the critical behavior is described by two *non-interacting* conformal theories associated with the two branches $\sigma = \pm$.

The excited states are obtained adding particles (holes) above (below) the Fermi level. Let us consider the following excitations: (i) we can add (remove) $I = I_+ + I_-$ particles above (below) the Fermi level, where I_σ ($\sigma = \pm$)

is the number of particles added (removed) in the branch σ ; and/or (ii) we can remove Q_σ particles in highest occupied level of the left/right Fermi point of the branch σ and add these particles in the lowest free levels of the right/left Fermi point of the same branch (see Table I). The finite-size corrections of these states can also be obtained by using the Euler-Maclaurin formula, as done for the ground state energy. The result obtained is the following

$$E^{\{I_\sigma\}, \{Q_\sigma\}} - E_0 = \sum_{\sigma=\pm} \frac{2\pi v_F^\sigma}{L} \left[\frac{1}{4} I_\sigma^2 + (Q_\sigma + \delta_\sigma/2)^2 \right] + o(L^{-2}). \quad (17)$$

where $\delta_\sigma = \text{mod} [\text{mod}(I_\sigma, 2) + \text{mod}(I, 2), 2]$ in the region I and $\delta_\sigma = 0$ in the region II (note that in this region $v_F^+ = 0$). We observe, again, from the above equation that the Hamiltonian (8) is described by two *non-interacting* conformal theories. Similar expressions also have been found in models with more than one gapless mode.^{39–43} Comparing Eqs. (2) and (17) we identify the scaling dimensions as

$$d_{I_\sigma, Q_\sigma}^\sigma = \frac{1}{4} I_\sigma^2 + (Q_\sigma + \delta_\sigma/2)^2. \quad (18)$$

The scaling dimensions above have a similar structure of the one of the Gaussian model.³ Note that $d_{0,1}^\sigma = x_\epsilon = 1$ corresponds to the scaling dimension of the energy operator, as in the XX chain.

III. ENTANGLEMENT ENTROPY

We are interested, now, in the finite-size corrections of the Rényi entropy, in the critical regions I and II. The leading finite-size corrections of Rényi entropy is related with the central charge, while the sub-leading corrections are governed by the exponent p_α .¹³ Below, we determine the central charge c and the exponent p_α by using the universal behavior of these corrections.

In the case of quadratic Hamiltonians, as the one considered in this work, the Rényi entropy can be evaluated through the correlation matrix method^{44,45} (see also Refs. 46 and 47). The main idea of this method is that the eigenvalues $\{\nu_j\}$ of the correlation matrix $C_{p,q} = \langle c_p^\dagger c_q \rangle$, $p, q = 1, \dots, \ell$; are related with the eigenvalues of the reduced density matrix ρ_A .^{44,45} Due to this fact, the Rényi entropy can be expressed in terms of the eigenvalues $\{\nu_j\}$ in the following form

$$S_\alpha(L, \ell) = \frac{1}{1-\alpha} \sum_{j=1}^{\ell} \ln [\nu_j^\alpha + (1-\nu_j)^\alpha]. \quad (19)$$

We will present data only for ℓ even due to the geometry of the two-leg ladder, although the odd sites also present the universal behavior predicted by the CFT.

The elements of the correlation matrix $C_{p,q}$ associated with the ground state wave function, can be easily evaluated, and are given by

$$C_{(2p,2q)} = \frac{1}{2L} \frac{\sin[(p-q)2\pi\rho^+] + \sin[(p-q)2\pi\rho^-]}{\sin[\frac{(p-q)\pi}{L}]},$$

$$C_{(2p-1,2q)} = -\frac{1}{2L} \sum_{k_F^+ < |k| \leq k_F^-} \frac{J_1 + J_2 \exp(-ik)}{\sqrt{J_1^2 + J_2^2 + 2J_1J_2\cos k}} e^{-ik(p-q)},$$

$$C_{(2p-1,2q-1)} = C_{(2p,2q)} \text{ and } C_{(2p-1,2q)} = C_{(2q,2p-1)}^*. \quad (20)$$

Similar expressions can also be obtained for the excited states.

In the next two subsections, we determine numerically the eigenvalues of the correlation matrix $C_{p,q}$ in order to obtain the Rényi entropy of the ground state as well as of the excited states.

A. Ground State

Let us consider first the Rényi entropy of the ground state. In Fig. 4(a), we present the von Neumann entropy S_1 , as function of ℓ for systems of size $L = 800$, $h = 0$, and three sets of couplings (J_1, J_2, J_3) . These sets of couplings illustrate the behavior of S_1 in the three distinct regions of the phase diagram depicted in Fig. 3. As we observed in this figure, in the gapped phase (region III) the von Neumann entropy tends to a constant, as expected. On the other hand, for the critical regions I and II, S_1 increases in agreement with the CFT prediction [Eq. (5)]. The black/blue lines in Fig. 4(a) are fits to our data using Eq. (5). The central charge c obtained through this fit is $c = 1.00001$ ($c_{eff} = 2.00021$) for the sets of couplings associated with the critical region II (I). Similar results were acquired for several other sets of coupling for these two distinct critical regions. These results show that for the region II, the low-energy physics is describe by a CFT with central charge $c = 1$, as we have already predicted [see Eq. (15)]. It is very interesting to note that in the region I, where we have two gapless modes (instead of one as in the region II), we obtain an effective central charge $c_{eff} = 2$. This result resembles to the one found for the finite-size corrections of the ground state energy [Eq. (15)]. However, unlike the finite-size corrections of the energies, the entropy does not depend on the sound velocities. Then, it is expected that *each gapless mode* contributes for the finite-size corrections of the entropy as Eq. (5). For this reason, we get $c_{eff} = 2c = 2$.

Another interesting feature of the von Neumann entropy in the region I, is that it presents an unexpected oscillation. In order to observe better these oscillations is convenient to define the following difference

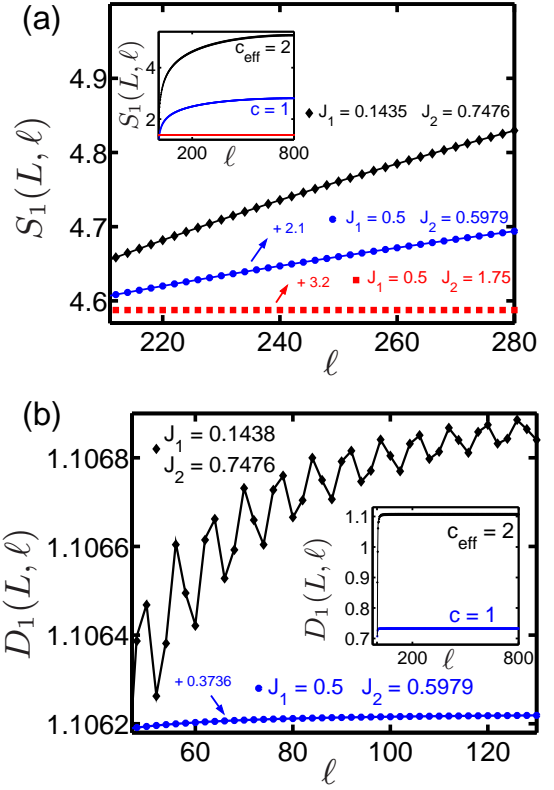


Figure 4: (Color online). (a) The von Neumann entropy $S_1(L, \ell)$ of the ground state for systems of size $L = 800$, $h = 0$, $J_3 = 1$ and some values of J_1 and J_2 (see legend). The couplings associate with the black, blue, and red data correspond to the couplings belonging to the regions I, II and III, respectively [see Fig. (3)]. (b) Results of the difference $D_1(L, \ell)$ for the same parameters of figure (a). Only few sites are presented. The solid lines in these figures connect the fitted points (see text). Insets: Show S_1 [figure (a)] and D_1 [figure (b)] for all sites. In order to show all data in the figures, we added some constants in the values of S_1 and D_1 , as indicated by the arrows.

$$D_\alpha(L, \ell) = S_\alpha(L, \ell) - \frac{c_{eff}}{6} \left(1 + \frac{1}{\alpha}\right) \ln \left[\frac{2L}{\pi} \sin \left(\frac{\pi \ell}{2L} \right) \right],$$

where $c_{eff} = 1(2)$ in the region II(I). From Eqs. (3)-(6), it is expected that the difference D_α behaves as

$$D_\alpha(L, \ell) = c_\alpha +$$

$$\frac{[a_1 \delta_{1,\alpha} + g_\alpha (1 - \delta_{1,\alpha}) \cos(\kappa \ell + \phi)]}{L^{p_\alpha}} \left| \sin \left(\frac{\pi \ell}{2L} \right) \right|^{-p_\alpha}. \quad (21)$$

In Fig. 4(b), we show $D_1(L, \ell)$ for the two sets of coupling parameters presented in Fig. 4(a). If $J_1 \neq J_2$, oscillations between the odd and even sites appear naturally in the entropy. The oscillation observed in Fig. 4(b), for the

set of coupling belonging to the Region I, is not related with the "dimerization", is present even for $J_1 = J_2$. According to a previous conjecture of Xavier and Alcaraz,¹⁶ based in studies of several models with just one gapless mode, the von Neumann entropy of single interval should not present these oscillations. Our present results, show that the same conjecture does not apply for models with more than one gapless mode. These authors also conjectured that the sub-leading finite-size correction of the von Neumann entropy decays with the exponent $p_1 = \nu = 2$. Indeed, we also have observed this decay in the region II. The blue solid line in Fig. 4(b) is the fit to our data assuming that $D_1(L, \ell)$ behaves as Eq. (21). The exponent that we get by this fit is $p_1 = 2.0001$, in agreement with the previous conjecture.

In the region I, where we have two gapless mode, we fit our data assuming that $D_1(L, \ell)$ has an oscillating term different of Eq. (6). In this region, we had to replace $g_1(1 - \delta_{1,\alpha}) \cos(\kappa\ell + \phi)$ in Eq. (6) by the following term: $g_1 \cos(\ell k_F^-) + g_1^{(2)} \cos(\ell k_F^+) + g_1^{(3)} \cos\left(\frac{k_F^- + k_F^+}{2}\ell\right) + g_1^{(4)} \cos\left(\frac{k_F^- - k_F^+}{2}\ell\right)$, in order to fit our data. By using this oscillating term, we are able to fit perfectly our data, as shown in Fig. 4(b). The exponent obtained in this later case is $p_1 = \nu = 2.0007$. Similar results were also observed for other couplings. The motivation for us consider the above oscillating term is given below. These results indicate that the sub-leading correction of the von Neumann entropy indeed has the exponent $p_1 = \nu = 2$ even for models with more than one gapless modes.

Let us now consider the α -Rényi entropies of the ground state with $\alpha > 1$. As illustration, we present in Fig. 5(a), $S_\alpha(L, \ell)$ as function of ℓ for systems of size $L = 800$ and $L = 40$, two values of α , and some couplings. In the case that $\alpha > 1$, it is expected oscillations in the entropy even for systems with PBC,¹³ as mentioned in the introduction. Moreover, it is expected that the amplitude of the oscillations decrease, as the system size increases [see Eq. (6)]. Indeed, this feature is clearly noted in a system with smaller size, as depicted in the inset of Fig. 5(a).

Once again, in order to observe better these oscillations we investigate the difference $D_\alpha(L, \ell)$. In Fig. 5(b), we show $D_\alpha(L, \ell)$ vs ℓ for the same data presented in Fig. 5(a) with $\alpha = 3$. Let us first discuss the data of the Rényi entropy in the region II. In this region, we are able to fit our data by using Eq. (6) with $\kappa = k_F^-$ and $\phi = 0$. The exponent we get from this fit is $p_3 = 0.666$. Similar fits of S_α (not shown) for $\alpha = 2$ and $\alpha = 4$ give $p_2 = 0.999$ and $p_4 = 0.497$, respectively. These results indicate that $p_\alpha = 2/\alpha$, as in the XX chain.

On the other hand, it is not possible to fit the data of the Rényi entropy, in the region I, if we consider that the oscillating term behaves just as $\cos(\kappa\ell + \phi)$, as it was done in the region II. In order to get some insight about these oscillations in the region I, we calculate analytically the spin-spin correlation function $\langle s_{2n}^z s_{2n+2\ell}^z \rangle$, which is

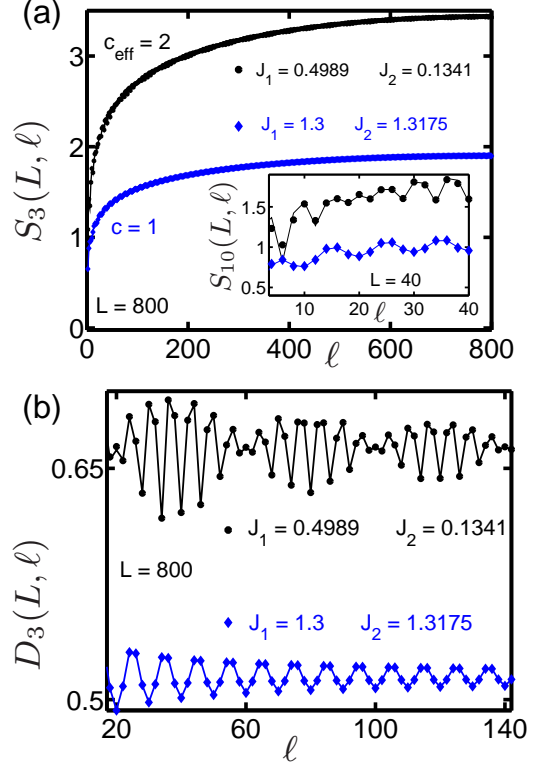


Figure 5: (Color online). (a) The Rényi entropy of the ground state for systems of size $L = 800$, $h = 0$, $J_3 = 1$, $\alpha = 3$, and two values of J_1 and J_2 (see legend). The couplings associate with the black and blue lines, correspond to couplings belonging to the regions I, and II, respectively [see Fig. (3)]. Inset: Results for $L = 40$. (b) The difference $D_\alpha(L, \ell)$ for the same parameters of figure (a). Only few sites are presented. The symbols in the figures (a) and (b) are the numerical data and the solid lines connect the fitted points (see text).

given by

$$\langle s_{2n}^z s_{2n+2\ell}^z \rangle = m^2 - \frac{1}{4L^2 \sin^2(\pi\ell/2L)} \left[1 - \frac{1}{2} \cos(\ell k_F^-) - \frac{1}{2} \cos(\ell k_F^+) - \cos\left(\frac{k_F^- + k_F^+}{2}\ell\right) + \cos\left(\frac{k_F^- - k_F^+}{2}\ell\right) \right].$$

Motivated by the fact that it is expected that the oscillations of the entropy are associated with the anti-ferromagnetic nature of the Hamiltonian,¹⁸ we assume that the difference $D_\alpha(L, \ell)$, in the region I, behaves as

$$D_\alpha(L, \ell) = c_\alpha + \left[a_1 \delta_{1,\alpha} + g_\alpha \cos(\ell k_F^-) + g_\alpha^{(2)} \cos(\ell k_F^+) + g_\alpha^{(3)} \cos\left(\frac{k_F^- + k_F^+}{2}\ell\right) + g_\alpha^{(4)} \cos\left(\frac{k_F^- - k_F^+}{2}\ell\right) \right] \left| L \sin\left(\frac{\pi\ell}{2L}\right) \right|^{-p_\alpha}. \quad (22)$$

$\left\{ \begin{array}{c} \circ \circ \mid \bullet \bullet \cdots \bullet \bullet \mid \circ \circ \\ \circ \circ \mid \bullet \bullet \cdots \bullet \bullet \mid \circ \circ \end{array} \right\}$	gs - Region (I)	
$\left\{ \begin{array}{c} \circ \circ \mid \circ \circ \cdots \circ \circ \mid \circ \circ \\ \circ \circ \mid \bullet \bullet \cdots \bullet \bullet \mid \circ \circ \end{array} \right\}$	gs - Region (II)	
Compact Excitations		
$\left\{ \begin{array}{c} \circ \circ \mid \circ \circ \cdots \circ \circ \mid \circ \circ \\ \circ \bullet \mid \bullet \bullet \cdots \bullet \bullet \mid \circ \bullet \end{array} \right\}$	$(:)(: \pm 1) -$	$I_- = 2$
$\left\{ \begin{array}{c} \circ \circ \mid \circ \circ \cdots \circ \circ \mid \circ \circ \\ \circ \bullet \mid \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \end{array} \right\}$	$(:)(: 1) -$	$I_- = 1$
$\left\{ \begin{array}{c} \circ \circ \mid \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \\ \circ \bullet \mid \circ \circ \cdots \circ \circ \mid \bullet \bullet \end{array} \right\}$	$(: 1) + (-1 :) -$	$I_{\pm} = \pm 1$
$\left\{ \begin{array}{c} \circ \bullet \mid \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \\ \bullet \bullet \mid \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \end{array} \right\}$	$(1, 2 :) + (: 1, 2) -$	$I_{\pm} = \mp 2 \quad Q_{\pm} = \mp 1$
Non-Compact Excitations		
$\left\{ \begin{array}{c} \circ \circ \mid \bullet \bullet \cdots \bullet \bullet \mid \circ \circ \\ \bullet \bullet \mid \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \end{array} \right\}$	$(:)(+ (1 : 1) -$	$j_- = 1$
$\left\{ \begin{array}{c} \circ \bullet \mid \circ \circ \cdots \circ \circ \mid \circ \circ \\ \bullet \bullet \mid \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \end{array} \right\}$	$(-1 : -1) \pm$	$j'_{\pm} = 1$
$\left\{ \begin{array}{c} \circ \circ \cdots \circ \circ \mid \circ \circ \mid \circ \circ \\ \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \mid \bullet \bullet \end{array} \right\}$	$(:)(+ (1, 2 : 1, 2) -$	$j_- = 4$
$\left\{ \begin{array}{c} \circ \bullet \cdots \circ \circ \mid \circ \circ \mid \circ \circ \\ \bullet \bullet \cdots \bullet \bullet \mid \bullet \bullet \mid \bullet \bullet \end{array} \right\}$	$(:)(+ (\pm 1 : 1) -$	$I_- = j_- = 1$

Table I: Schematic representations of some states. The open (closed) cycles are the empty (occupied) levels. The two rows of cycles inside the symbol $\{\}$ denotes the two branches, while the vertical lines show the positions of the Fermi momenta. We also present the excitations in terms of I_{σ} and Q_{σ} [see Eq. (17)], and the notation used by the authors of Ref. 24. In the later case $(h_1 h_2 \dots : p_1 p_2 \dots)_{\sigma}$ represents an excited state with holes (particles) in the h_i s (p_i s) allowed momentum values below (above) the Fermi point of the branch σ .

Indeed, as illustrated in Fig. 5(b), we obtain a very nice fit of our data if we assume that D_{α} behaves as the above equation. The exponent we get from the fit is $p_3 = 0.667$.

Similar results, as the ones presented above, were obtained for several other values of m and coupling parameters. In addition to that, the results for some values of α strongly support that the exponent $p_{\alpha} = 2/\alpha$, as happens in the XX chain.¹³ These results are in agreement with the conjecture¹⁶ that the exponent p_{α} is related with the dimension of the energy operator x_{ϵ} by $p_{\alpha} = 2x_{\epsilon}/\alpha$, which is $x_{\epsilon} = 1$ in the present model.

B. Excited states

Finally, we consider the Rényi entropy of the excited states. Differently of the Rényi entropy of the ground state, where several studies indeed confirmed the universal behavior predicted by the CFT, few works considered the excited states.^{9,24–26,48–51} Certainly, more studies are highly desired in this front and in this subsection we intend to provide a study of the Rényi entropy of several excited states of the Hamiltonian (8).

First, we consider the Rényi entropy associated with compact excitations. The compact excitations are those that do not present holes in the momentum space. In

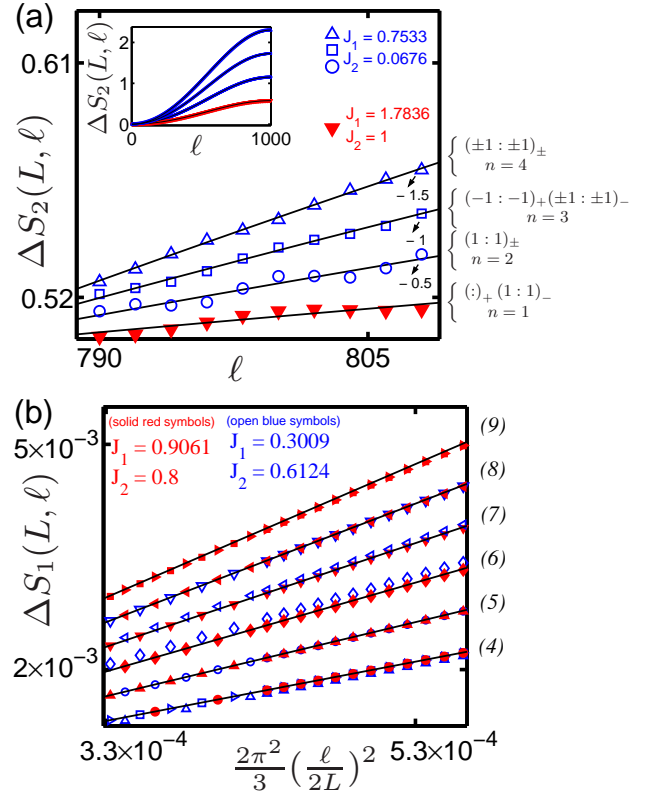


Figure 6: (Color online). (a) ΔS_2 vs ℓ for some particle-hole excitations for systems of size $L = 1000$, $h = 0$, and some couplings (see legend). The symbols are the numerical data and the solid lines connect the fitted data (see text). The solid curves from the top to down have values of n : 4, 3, 2 and 1 [see Eq. 23]. The blue (red) symbols are data from the region I (II). Only few sites are presented. In order to show all data in the figure, we added some constants in the values of ΔS_2 , as indicated by the arrows. We also present the excitations in terms of the notation of Ref. 24. Inset: ΔS_2 for all sites. (b) ΔS_1 vs $\frac{2\pi^2}{3}(\ell/2L)^2$ for several excitations for systems of size $L = 10000$ (see Table II).

Table I, we depict few examples of such excitations. We found that $\Delta S_{\alpha}(L, \ell) = S_{\alpha}^{exc} - S_{\alpha}^{gs} \lesssim 10^{-3}$ for these compact excitations, as well as many others not presented in this table. These results are in agreement with the prediction, done by the authors of the Refs. 9 and 24, that $S_{\alpha}^{exc} = S_{\alpha}^{gs}$ for the compact excitations.

The Rényi entropies associated with the non-compact excitations are, in general, different from the ground state. Let us consider particle-hole excitations in the right/left Fermi points of the branch σ . In Fig. 6(a), we present ΔS_2 as function of ℓ for some these excitations for systems with $2L$ sites. As observed in this figure, the Rényi entropies associated with these non-compact excitations increase with ℓ . Moreover our results, show that the ΔS_2 data can be perfectly fitted (except by the unusual oscillations) if we assume that [see Eq.(7)]

$$\Delta S_2 = -nF_{i\partial\phi}^{(2)} = -n \ln(1 - 2s^2 + 3s^4 - 2s^6 + 8s^8), \quad (23)$$

where $s = \sin(\pi\ell/4L)$ and n is the number of particle-hole excitations associated with the excited state.

In the case of just *one gapless* mode, the Eq. (23) was determined analytically for the free boson theory.²⁴ Note that for a free boson theory with *two decoupled field* ϕ^σ ($\sigma = \pm$) the Eq. (23) is valid.

Finally, we consider $\Delta S_1(L, \ell) = S_1^{exc} - S_1^{gs}$ for several excitations. In particular we focus on the behavior of ΔS_1 in the regime that $(\ell/L) \ll 1$. In this regime it is expected that ΔS_1 , for a system with $2L$ sites, behaves as^{9,24}

$$\Delta S_1 = \frac{2\pi^2}{3} d^\beta \left(\frac{\ell}{2L} \right)^2 + o[(\ell/L)^2], \quad (24)$$

where d^β is the scaling dimension of the operator associate with the excitation. This behavior of ΔS_1 above was proofed⁹ in the case of the primary operator $i\partial\phi$ and verified numerically for the descendant operators.²⁴

The scaling dimensions of the XX chain have the same structure of the Gaussian model³ (with compactification radius $R = 2$) and are given by $d_{I,Q,j,j'} = x_{I,Q} + j + j'$, where $x_{I,Q} = (\frac{I^2}{4} + Q^2)$ are the scaling dimensions of the primary operators and $j, j' = 0, 1, 2, \dots$ [see Eq. (2)]. Note that the energy associated with an operator with dimension $d_{I,Q,j,j'}$ (in the Gaussian model) has momentum $\Delta P_{I,Q}^{j,j'} = \frac{2\pi}{L}(IQ + j - j')$.³ We also found a similar result. The momentum of a state associated with the scaling dimension $\sum_{\sigma=\pm}(d_{I_\sigma,Q_\sigma}^\sigma + j_\sigma + j'_\sigma)$ is given by $\Delta P = \sum_{\sigma=\pm} \Delta P_{I_\sigma,Q_\sigma,j_\sigma,j'_\sigma}^\sigma$, where $\Delta P_{I_\sigma,Q_\sigma,j_\sigma,j'_\sigma}^\sigma = \frac{2\pi}{L}[I_\sigma(Q_\sigma + \delta_\sigma/2) + j_\sigma - j'_\sigma]$. The momenta of the configurations presented in Table II are in perfect agreement with the values of ΔP .

In Fig. 6(b), we show ΔS_1 as function of $\frac{2\pi^2}{3}(\ell/2L)^2$ for several excitations, in the regime that $(\ell/L) \ll 1$ for a system with $2L$ sites. The schematic representation of these non-compact excitations are also presented in terms of I_σ, Q_σ and j_σ in Table II. The solid lines in Fig. 6(b) correspond to Eq. (24) with the values of d^β shown in parentheses. As observed in this figure, the values of d^β that fit our data to Eq. (24) are consistent with $d^\beta = \sum_{\sigma=\pm}(j_\sigma + j'_\sigma)$. Slightly different than expected from the Eq. (24). In order to better understand this discrepancy, we also reproduce the results of the same excited states considered in the Refs. 9 and 24 for the XX chain. Indeed, we also get the same values of d^β obtained in these references. For example, for the excitation associated with the operator $i\partial\phi$, i. e. $(1:1)$, we get $d^\beta = 1$. However, we did not interpret this value as the scaling dimension of the *primary* operator $i\partial\phi$ (which is 1), due to the momentum of the excitation $(1:1)$. From the CFT point of view this primary operator has momentum 0. However, the excitation $(1:1)$ has momentum $\Delta P = 2\pi/L$. These results show that a deeper analysis

		REGION II					
		I_-	Q_-	j_-			
●	$\left\{ \begin{array}{c c} \circ & \circ$						

Table II: Schematic representations of the excited states considered in Fig. 6(b). We also present the excitations in terms of I_σ and Q_σ [see Eq. (17)] as well as the notation of Ref. 24.

of the Rényi entropy of excited states associated with the descendent operators should be done in order to understand the exact dependence of ΔS_1 , in the regime that $(\ell/L) \ll 1$. Below, we comment on two points that may help infer the exact form of the factor d^β that appears in (24). First, note that Eq. (24) was obtained exactly for the primary operators. For instance, the operator $i\partial\phi$ is primary in terms of the Virasoro algebra. However, for a larger algebra as the Kac-Moody algebra (which is the correct one for the XXZ chain) the operator $i\partial\phi$ can be seen as a descendent operator. So, the result of Eq. (24) independent whether the operator $i\partial\phi$ is primary or not. Second, our results suggest that the operators associated with configurations as the ones depicted in Table II may be express by a product of vertex operators with scaling dimensions $d_{I,Q}$ times a product of derivatives of the fields ϕ and $\bar{\phi}$ with scaling dimension $j + j'$, and this would lead a factor $d^\beta = j + j'$ in Eq. (24).⁵² Further investigations are needed in order to clarify the exact form of ΔS_1 in the limit $(\ell/L) \rightarrow 0$ for a general excited state.

IV. CONCLUSIONS

In this work, we considered a toy model with three spin-interactions defined on a two-leg geometry which is exactly solvable by means of a Jordan-Wigner transformation. We were able to obtain analytically the finite-size corrections of the low-lying energies, and for this reason we get the exact values of the central charge and the scaling dimensions. The toy model investigated [Eq. (8)] is in the same universality class of critical behavior of the XX chain, with central charge $c = 1$. Moreover, the scaling dimensions of the toy model present a similar structure as the one of the XX chain [see Eq. (18)].

We also acquired the central charge by analyzing the finite-size corrections of the Rényi entropy. In the region of the phase diagram where there is one gapless mode, we get $c = 1$. Whereas in the region where there are two gapless modes, we found that Rényi entropy behaves asymptotically as

$$S_\alpha(L, \ell) = \frac{c_{eff}}{6} \left(1 + \frac{1}{\alpha}\right) \ln \left[\frac{L}{\pi} \sin \left(\frac{\pi \ell}{L} \right) \right] + a_\alpha, \quad (25)$$

where $c_{eff} = 2c = 2$. For systems with n_{gl} gapless modes it is expected that $c_{eff} = n_{gl}c$, at least for non-interaction systems (see also Fagotti in Ref. 34 for a simi-

lar discussion). It is interesting to mention that an extension of the present model for N -leg ladders with $(N+1)$ spin interactions can also be done and would lead to an effective central charge $c_{eff} = N$ (at least for some regions of the parameter space). This result shows that the von Neumann entropy of the N -leg spin ladders of size L and L gapless modes should behave as $S_1 \sim L \ln(L) + aL$, this simply argument shows that entropic area law may be violated in two-dimensional critical systems. Indeed, violations of the entropic area law was observed in systems in dimension higher than one^{53,54}

We also study the Rényi entropy of the excited states. Our results of the Rényi entropy associated with the compacts as well as the non-compact excitations are in agreement with the recent prediction done by the authors of Refs. 9 and 24.

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